

Dimensionality as a Perturbation Parameter in the
Generalized Hydrogen Atom

by

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Abstract

A recent suggestion has been made that the hydrogen bound state spectrum should not depend on the number of spatial dimensions. It is pointed out here that the uncertainty principle implies that such differences must exist and that a perturbation expansion in the dimensionality parameter yields a precise quantitative confirmation of the effect.

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The solution of the Schrodinger equation for the hydrogen atom in other than three dimensions has been the object of much study in recent decades. In particular the bound state spectrum has been found [1] in the two dimensional case to be of the form

$$E_n = -\frac{Me^4}{2\hbar^2(n-1/2)^2} \quad n = 1, 2, \dots \quad (1)$$

where M and e are respectively the mass and charge of the electron and n is the principal quantum number. Subsequently the result (1) has been rederived [2-8] in a number of different ways. The most striking feature of (1) is perhaps the fact that because of the occurrence of the $n - 1/2$ factor it predicts a two-dimensional ground state energy which is four times that of the corresponding three-dimensional one.

Despite the considerable familiarity achieved by the result (1), it is far from universally known. Thus, for example, a recent work [9] in which (1) has been derived yet again proposes to require that it reproduce the three-dimensional result. Since n is linear in the angular momentum quantum number m , this can be achieved if m (and thus n) is required to be half-integral. On the other hand this implies the existence of wave functions which are not single-valued when m assumes such noninteger values.

Since the literature on this problem is extensive, there is no need here for one more derivation of the result (1). Suffice it to say that there is no real motivation for the half integral angular momentum hypothesis to bring the two and three dimensional results into agreement. On the other hand ref. 9 does pose a somewhat interesting issue in suggesting that the presence of a third dimension on physical grounds should not affect the solution. Such a claim is certainly correct at the classical level where the three dimensional central field problem can always be confined to consideration of motion in a plane.

It is clear, however, that quantum mechanical considerations do not allow this simple picture, which would only be valid if one could simultaneously require both the momentum and coordinate associated with the third dimension to vanish. This, of course, suggests that the uncertainty principle might profitably be used to clarify the physical picture.

A highly simplified invocation of the latter could proceed in the following way. One can imagine that the third coordinate enters the problem in a fairly trivial manner by including only its contribution $p_z^2/2M$ to the Hamiltonian. This could be a realistic picture for orbits which are essentially circular and correspond to large radii. In this idealization the energy levels can be expected to be raised in the three dimensional case relative to the two dimensional one by an amount of the order of $p_z^2/2M$ where the magnitude of p_z is estimated by the uncertainty principle to be

$$|p_z| \sim \hbar/a_0$$

where $a_0 = \hbar^2/Me^2$ is the Bohr radius. This is readily seen to raise the original two dimensional value by an amount of the order of n^{-2} Rydberg units, which actually provides a very reasonable approximation to the energy shift associated with the third dimension.

It is actually possible to improve considerably this crude calculation, and in fact that is the principal aim of this paper. Also it is of interest to note that since Nieto [5] has given the bound state spectrum of the N dimensional hydrogen atom

$$E_n = -\frac{Me^4}{2\hbar^2} \frac{1}{[n + 1/2(N-3)]^2}, \quad (2)$$

it is possible to establish an even more general result.

One begins with the Hamiltonian

$$H = \frac{1}{2M} p^2 - \frac{e^2}{r} \quad (3)$$

where

$$p^2 = \sum_{i=1}^N p_i^2$$

$$r^2 = \sum_{i=1}^N x_i^2$$

and it is assumed that $N \geq 2$. The concern here is principally with the additional dimensions $N > 2$. Thus one writes

$$\frac{1}{r} = (x_1^2 + x_2^2)^{-1/2} - \frac{1}{2}(x_1^2 + x_2^2)^{-3/2} \sum_{i=3}^N x_i^2 + \dots$$

This can be expected to provide a valid basis for an expansion of the N dimensional hydrogen energy levels in terms of the $N = 2$ spectrum provided that one restricts consideration to orbits for which $x_1^2 + x_2^2$ is large and essentially constant (i.e., circular orbits with $n \rightarrow \infty$).

Thus the Hamiltonian (3) is approximated by

$$H = H_0 + H_1$$

where H_0 has the $N = 2$ hydrogen atom form and

$$H_1 = \sum_{i=3}^N \left(\frac{1}{2M} p_i^2 + \frac{1}{2} e^2 \langle (x_1^2 + x_2^2)^{-3/2} \rangle x_i^2 \right) .$$

Evidently the correction to the ground state $N = 2$ energy is simply the energy associated with $N - 2$ harmonic oscillators. Thus the minimum correction to the $N = 2$ spectrum is given by

$$E_n = -\frac{Me^4}{2\hbar^2(n-1/2)^2} + \frac{1}{2}(N-2)\hbar\omega \quad (4)$$

where

$$\omega^2 = \frac{e^2}{M} \langle (x_1^2 + x_2^2)^{-3/2} \rangle .$$

From the corresponding three dimensional result one finds readily that for $N = 2$ in a state of orbital angular momentum m and principal quantum number n

$$\langle (x_1^2 + x_2^2)^{-3/2} \rangle_{n,m} = a_0^{-3} [(n-1/2)^3 |m|(m^2 - 1/4)]^{-1}$$

so that in the most nearly circular orbit states (i.e., $n = |m| + 1$)

$$\omega = \frac{Me^4}{\hbar^3} \frac{1}{(n-1/2)^2} \frac{1}{[(n-1)(n-3/2)]^{1/2}} . \quad (5)$$

Upon expanding the exact result (2) to first order in the parameter $N - 2$ one obtains for the energy shift associated with the $N - 2$ additional dimensions

$$\Delta E_n = \frac{Me^4}{2\hbar^2} \frac{N-2}{(n-1/2)^3} . \quad (6)$$

On the other hand (4) and (5) yield for this quantity

$$\Delta E_n = \frac{Me^4}{2\hbar^2} \frac{N-2}{(n-1/2)^2} \frac{1}{[(n-1)(n-3/2)]^{1/2}}$$

which clearly reproduces (6) in the large n limit.

The calculation presented here has succeeded in reproducing exactly the minimum (positive) energy contribution associated with each additional spatial dimension of the generalized hydrogen atom. This has required (as expected) that the orbits be large and essentially circular so that the coefficient of x_i^2 in the expansion of the $1/r$ potential can be made as small and as nearly constant as possible. As a result of this fairly elementary exercise one can also better appreciate what might otherwise be considered a fairly odd mathematical fact – namely, that the ground state of hydrogen for $N = 2$ is four times that of the $N = 3$ case. Specifically, in the latter case a certain positive amount of energy must be added (i.e., the energy must be raised) merely to accomplish even a minimal localization in the third coordinate.

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